Configuration Spaces and $\mathbb{R}^n$

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September 16, 2004

This note attempts to make clear the relation between configurations of points in a space Y and those in its Cartesian product with the reals. It turns out to be a very simple relation whose proof uses nothing new.

Let Y be an unbased space. Denote by Y^j the j-fold Cartesian product of Y with itself. For present purposes we consider the circle S^1 to be the quotient of the unit interval $[0,1]/\{0,1\}$. If X is a based space then ΣX is defined to be $X \wedge S^1$ and ΩX is defined to be the loop space of X, that is, the space of based maps from S^1 to X.

Definition 1 *Define $\mathscr{C}(Y)_j$ to be the subspace of Y^j consisting of j-tuples of distinct points in Y. If ν is an injective function from $\{1,\ldots,i\}$ to $\{1,\ldots,j\}$ then define $\nu^* : \mathscr{C}(Y)_j \to \mathscr{C}(Y)_i$ by sending $(y_1,\ldots,y_i)$ to $(y_{\nu(1)},\ldots,y_{\nu(i)})$. If X is a nondegenerately based space, define $\nu_* : X^i \to X^j$ sending $(x_1,\ldots,x_i)$ to $(x'_1,\ldots,x'_j)$ where $x'_k = x_l$ if $k=\nu(l)$ and x'_k is the basepoint if k is not in the image of ν.*

Note that these maps are compatible with composition; i.e. $(\nu\circ\mu)_* = \nu_*\circ\mu_*$ and $(\nu\circ\mu)^* = \mu^*\circ\nu^*$. In particular, the maps ν^* define a free action of the j-fold symmetric group $\mathcal{S}_j$ on $\mathscr{C}(Y)_j$.

The spaces $\mathscr{C}(Y)_j$ and the maps ν^* define a *coefficient system* in the sense of [2], and we define an equivalence relation $\sim$ on $\coprod_j \mathscr{C}(Y)_j \times X^j$ generated by $(\nu^*(\vec{y}),\vec{x}) \sim (\vec{y},\nu_*(\vec{x}))$. Define

$$C(Y,X) = \left(\coprod_j \mathscr{C}(Y)_j \times X^j\right) \Big/ \sim .$$

In their recent paper [5], Cohen and Taylor deal with the space $C(\mathbb{R}\times Y, X)$. Recall that a *weak metric space* is a space Y together with a continuous function $d : Y\times Y \to [0,\infty)$ such that $d^{-1}(0)$ is the diagonal in $Y\times Y$. The main result of this note is:

Theorem 1 *Let Y be a weak metric space and X a nondegenerately based space. There is a space $C_1(Y,X)$ and a pair of maps*

$$C(\mathbb{R}\times Y, X) \xleftarrow{\phi} C_1(Y,X) \xrightarrow{\alpha} \Omega C(Y, \Sigma X)$$

such that:

1. *$C_1(-,-)$ is functorial with respect to based maps in the second variable and injective maps in the first variable, and ϕ and α are natural;*
2. *ϕ is a homotopy equivalence; and*
3. *α is a weak homotopy equivalence if X is path-connected.*

The proof uses the methods from [1] and [2]. The space $C_1(Y,X)$ is another space derived from a "coefficient system." Let $\mathscr{C}_1(Y)_j$ be the subspace of $(\mathbb{R}\times\mathbb{R}\times Y)^j$ consisting of j-tuples of triples $\big((a_1,b_1,y_1),\dots,(a_j,b_j,y_j)\big)$ such that for all i, $a_i<b_i$ and for all $k\neq l$, $y_k=y_l$ implies $b_k\le a_l$ or $b_l\le a_k$.

We can define a coefficient system structure ν^*, ν_* on $\{\mathscr{C}_1(Y)_j\}_{j\ge 0}$ by acting on triples, and define and $\sim$ on $\coprod_j \mathscr{C}_1(Y)_j\times X^j$ generated by $(\nu^*(\kappa),\vec{x})\sim(\kappa,\nu_*(\vec{x}))$. The quotient space $C_1(Y,X)$ can be thought of as consisting of configurations of line segments in $\mathbb{R}\times Y$ with disjoint interiors, labeled by points of X; a segment labeled by the basepoint drops out under the identification $\sim$. For compactness of notation, we will use

$$\big(a_i,b_i,y_i\big)_{1\le i\le j}$$

as shorthand for $\big((a_1,b_1,y_1),\dots,(a_j,b_j,y_j)\big)\in\mathscr{C}_1(Y)_j$, and

$$\big[a_i,b_i,y_i,x_i\big]_{1\le i\le j}$$

for the image of $\big((a_1,b_1,y_1),\dots,(a_j,b_j,y_j),(x_1,\dots,x_j)\big)$ in $C_1(Y,X)$. Similarly we will use the shorthand $\big[y_i,x_i\big]_{1\le i\le j}$ for points of $C(Y,X)$.

There is an obvious map ϕ_j from $\mathscr{C}_1(Y)_j$ to $\mathscr{C}(\mathbb{R}\times Y)_j$ taking each segment to its center-point. This map respects permutations and so induces a map ϕ from $C_1(Y,X)$ to $C(\mathbb{R}\times Y,X)$.

There is also a map $\bar{\phi}_j$ from $\mathscr{C}(\mathbb{R}\times Y)_j$ to $\mathscr{C}_1(Y)_j$ which we define as follows. Use the weak metric d on Y to define $g:(\mathbb{R}\times Y)\times(\mathbb{R}\times Y)\to[0,\infty)$ by setting

$$g\big((a,y),(a',y')\big)=\frac{1}{2}\left(\frac{|a-a'|^2+d(y,y')}{|a-a'|+d(y,y')+1}\right)$$

so $g((a,y),(a',y'))\le\frac{1}{2}|a-a'|$ if $y=y'$. Let $\kappa=((a_1,y_1),\dots,(a_j,y_j))\in\mathscr{C}(\mathbb{R}\times Y)_j$ and define

$$v(\kappa)=\min_{k\neq l}\big\{g((a_k,y_k),(a_l,y_l))\big\}.$$

It's clear that $v(\kappa) > 0$ and that the intervals $[a_k - v(\kappa), a_k + v(\kappa)]$ and $[a_l - v(\kappa), a_l + v(\kappa)]$ do not overlap when $y_k = y_l$, so we can define

$$\bar{\phi}_j(\kappa) = \big(a_i - v(\kappa), a_i + v(\kappa), y_i\big)_{1\le i\le j}.$$

These induce a map $\bar{\phi} : C(\mathbb{R}\times Y, X) \to C_1(Y,X)$. Further, ϕ_j and $\bar{\phi}_j$ are easily seen to be inverse $\mathcal{S}_j$-equivariant homotopy equivalences: $\phi_j\bar{\phi}_j$ is the identity of $\mathscr{C}(\mathbb{R}\times Y)_j$, and there is a deformation from the identity of $\mathscr{C}_1(Y)_j$ to $\bar{\phi}_j\phi_j$ by linearly scaling the intervals around their centers. So by Lemma 2.7(ii) of [2], ϕ is a homotopy equivalence.

Next we need to define α. For the purposes of this section it is more convenient to work with a homeomorphic copy of $C_1(Y,X)$. Let

$$\bar{C}_1(Y,X) = \Big\{\big[a_i, b_i, y_i, x_i\big]_{1\le i\le j} \in C_1(Y,X)\big| 0 < a_i < b_i < 1 \text{ for all } i\Big\}$$

This subspace is clearly homeomorphic to $C_1(Y,X)$ via the homeomorphism of the reals $\mathbb{R}$ with the open interval $(0,1)$. Let $w = [a_i, b_i, y_i, x_i]_{1\le i\le j}$ be a point of $\bar{C}_1(Y,X)$. For a given t, define

$$\alpha(w)(t) = \big[y_i, [x_i, s_i]\big]_{1\le i\le j \text{ and } a_i\le t\le b_i},$$

where $s_i = (t-a_i)/(b_i-a_i)$. For a given t and for each i satisfying $a_i \le t \le b_i$, we observe that $0 \le s_i \le 1$ and the points $\big\{y_i \big| 1\le i\le j \text{ and } a_i\le t\le b_i\big\}$ are distinct; also $\alpha(w)(0) = \alpha(w)(1)$ is the basepoint $*$ of $C(Y,\Sigma X)$. Thus $\alpha(w)$ is a well-defined loop in $C(Y,\Sigma X)$.

To show α is a weak equivalence, we use the same idea as [1], namely to fit it into a comparison of quasifibration sequences. Define $E_1(Y,X)$ to be the quotient space of $\bar{C}_1(Y,X)\times[0,1]$ where we identify $\Big(\big[a_i, b_i, y_i, x_i\big]_{1\le i\le j}, s\Big)$ and $\Big(\big[a_i', b_i', y_i', x_i'\big]_{1\le i\le j+k}, s\Big)$ if $(a_i, b_i, y_i, x_i) = (a_i', b_i', y_i', x_i')$ for $1\le i\le j$ and $a_{j+1},\ldots,a_{j+k} \ge s$. Note that all points of the form $(w,0)$ are identified with the basepoint $(*,0)$ of $E_1(Y,X)$, so $E_1(Y,X)$ is contractible.

Define a map $\bar{\alpha}$ from $E_1(Y,X)$ to the path space $PC(Y,\Sigma X)$ by

$$\bar{\alpha}(w,s)(t) = \begin{cases} \alpha(w)(t), & \text{if } t\le s, \text{ and}\\ \alpha(w)(s), & \text{if } t\ge s. \end{cases}$$

Defining $\iota : \bar{C}_1(Y,X) \to E_1(Y,X)$ by $\iota(w) = (w,1)$ and $q : E_1(Y,X) \to C(Y,\Sigma X)$ by $q(w,s) = \bar{\alpha}(w,s)(1)$, we have the following commutative diagram

$$\begin{tikzcd}
\bar{C}_1(Y,X) \arrow[r, "\alpha"] \arrow[d, "\iota"'] & \Omega C(Y,\Sigma X) \arrow[d, hook] \\
E_1(Y,X) \arrow[r, "\bar{\alpha}"] \arrow[d, "q"'] & PC(Y,\Sigma X) \arrow[d, "p_1"] \\
C(Y,\Sigma X) \arrow[r, equal] & C(Y,\Sigma X)
\end{tikzcd}$$

where p_1 is projection on the endpoint. Thus by comparison of the long exact sequences of homotopy groups, it is enough to show that q is a *quasifibration*, that is, a map $q : E \to B$ such that for all $b \in B$ the canonical map from $q^{-1}(b)$ to the homotopy fiber of q over b is a weak homotopy equivalence.

Recall from [3] the Dold-Thom criterion for a map over a filtered base space to be a quasifibration. Let B be a space with closed subspaces

$$F_0B \subseteq F_1B \subseteq \ldots F_jB \subseteq \ldots \subseteq B$$

and $B = \bigcup_{j\geq 0} F_j$, and let $q : E \to B$ be a map. A subspace $V \subseteq B$ is called *distinguished* if the restriction $q : q^{-1}(V) \to V$ is a quasifibration. Then

Theorem 2 *(Dold and Thom) B is distinguished provided that*

1. *F_0B is distinguished, and for each $j > 0$ every open subset of $F_jB \smallsetminus F_{j-1}B$ is distinguished, and*
2. *for each $j > 0$ there is a homotopy $h_t : U \to U$ of a neighborhood U of $F_{j-1}B$ in F_jB, and a homotopy $H_t : q^{-1}(U) \to q^{-1}(U)$ such that:*
   - *(a) h_0 is the identity map of U, $h_1(U) \subseteq F_{j-1}B$, and for all t, $h_t(F_{j-1}B) \subseteq F_{j-1}B$,*
   - *(b) H_0 is the identity map of $q^{-1}(U)$ and for all t, $qH_t = h_tq$, and*
   - *(c) for all $z \in U$, the map $H_1 : q^{-1}(z) \to q^{-1}(h_1(z))$ is a homotopy equivalence.*

Here we give $C(Y, \Sigma X)$ the filtration of [1], that is $F_jC(Y, \Sigma X)$ is defined to be the image of $(\coprod_{0\leq k\leq j} \mathscr{C}(Y)_k \times (\Sigma X)^k)$. This has the property that $F_0C(Y, \Sigma X)$ consists of just the basepoint $*$, and $F_jC(Y, \Sigma X) \smallsetminus F_{j-1}C(Y, \Sigma X)$ is homeomorphic to the image of $\mathscr{C}_1(Y)_j \times \big(X \smallsetminus \{*\} \times (0, 1)\big)^j$.

We define some maps on $\bar{C}_1(Y, X)$ to help elucidate the proof. If $w = \big[a_i, b_i, y_i, x_i\big]_{1\leq i\leq j}$ and $w' = \big[a_i, b_i, y_i, x_i\big]_{j+1\leq i\leq j+k}$ are configurations in which for all $k \neq l$ the sets

$$\big\{(t, y_i) \in \mathbb{R} \times Y \big| a_i < t < b_i\big\}$$

are pairwise disjoint, then let $w \cup w' = \big[a_i, b_i, y_i, x_i\big]_{1\leq i\leq j+k}$. This is continuous on the subspace of $\bar{C}_1(Y, X) \times \bar{C}_1(Y, X)$ on which it is defined.

If s and t are real numbers with $s < t$ and $w = \big[a_i, b_i, y_i, x_i\big]_{1\leq i\leq j}$, then define

$$shrink_{s,t}(w) = \big[s + (t - s)a_i, s + (t - s)b_i, y_i, x_i\big]_{1\leq i\leq j},$$

which linearly compresses a configuration of segments in $(0, 1) \times Y$ into the slice $(s, t) \times Y$. Note that the composition $\mu : \bar{C}_1(Y, X) \times \bar{C}_1(Y, X) \to \bar{C}_1(Y, X)$ defined by

$$\mu(w, w') = shrink_{0,\frac{1}{2}}(w) \cup shrink_{\frac{1}{2},1}(w')$$

defines an H-space structure on $\bar{C}_1(Y,X)$.

For an element $z = \big[y_i, [x_i, s_i]\big]_{1\le i\le j} \in F_jC(Y,\Sigma X) \setminus F_{j-1}C(Y,\Sigma X)$ we define

$$\lambda(z) = \left[\frac{1}{2} - \frac{s_i}{2}, 1 - \frac{s_i}{2}, y_i, x_i\right]_{1\le i\le j}.$$

This maps via α to a loop whose value is z at $t = \frac{1}{2}$, and is well-defined and continuous on $F_jC(Y,\Sigma X) \setminus F_{j-1}C(Y,\Sigma X)$.

For an element $w \in \bar{C}_1(Y,X)$ and $s \in [0,1]$, we can define a function

$$below_s(w) = \big[a_i, b_i, y_i, x_i\big]_{1\le i\le j \text{ and } b_i \le s},$$

the segments of w contained in $[0,s]\times Y$. This is continuous on $q^{-1}(F_jC(Y,\Sigma X)\setminus F_{j-1}C(Y,\Sigma X))$.

For a relatively open set $V \subseteq F_jC(Y,\Sigma X)\setminus F_{j-1}C(Y,\Sigma X)$ define $\psi : \bar{C}_1(Y,X)\times V \to q^{-1}(V)$ by

$$\psi(w,z) = \left(shrink_{0,\frac{1}{2}}(w) \cup shrink_{\frac{1}{2},1}(\lambda(z)), \frac{3}{4}\right).$$

If $(w,s) \in E_1(Y,X)$ define $\bar{\psi}(w,s) = below_s(w)$. It follows that there is a commutative diagram

$$\begin{tikzcd}
\bar{C}_1(Y,X)\times V \arrow[rr, shift right, "\psi"'] \arrow[dr, "p_2"'] & & q^{-1}(V) \arrow[ll, shift right, "(\bar{\psi},q)"'] \arrow[dl, "q"] \\
& V &
\end{tikzcd}$$

The left map is projection on the second factor and so is the simplest kind of quasifibration; thus the proof of part (1.) will be complete when we have shown that ψ and $(\bar{\psi},q)$ are inverse equivalences over V. But this is clear: $\bar{\psi}\psi(w,z)$ is just $shrink_{0,\frac{1}{2}}(w)$, and if $q(w,s) = z$, then

$$\begin{aligned}
\psi(\bar{\psi}(w,s), q(w,s)) &= \psi(below_s(w), z) \\
&= \left((shrink_{0,\frac{1}{2}}(below_s(w)) \cup shrink_{\frac{1}{2},1}(\lambda(z)), \frac{3}{4}\right),
\end{aligned}$$

and linearly deforming all the segments to their original locations, and simultaneously deforming $\frac{3}{4}$ to s linearly, describes a homotopy over V of $\psi\circ(\bar{\psi},q)$ to the identity.

The proof of part (2.) rests on the fact that the inclusion $F_{j-1}C(Y,\Sigma X) \hookrightarrow F_jC(Y,\Sigma X)$ is a cofibration, which comes from the fact that X is nondegenerately based. Let W be a neighborhood of the basepoint $*$ in X and let $K_t : X \to X$ be a based homotopy where $K_0 = id$ and $K_1(W) = \{*\}$. Let L_t be a linear deformation of $[0,1]$ from the identity to the map

$$L_1(t) = \begin{cases} 0, & \text{if } t \le \frac{1}{4}; \\ 2t - \frac{1}{2}, & \text{if } \frac{1}{4} \le t \le \frac{3}{4}; \text{ and} \\ 1, & \text{if } t \ge \frac{3}{4}. \end{cases}$$

Use the same symbol L_t to denote the induced homotopy on S^1. Then $J_t = K_t \wedge L_t$ is a deformation of $\Sigma X = X \wedge S^1$ which collapses a neighborhood $W' = W \wedge \big([0, \frac{1}{4}) \cup (\frac{3}{4}, 1]\big)$ of the basepoint. Thus let

$$U = \left\{ \big[y_i, [x_i, s_i]\big]_{1 \leq i \leq j} \big| [x_i, s_i] \in W' \text{ for some } i \right\},$$

and use the functoriality of $C_1(-,-)$ to define $h_t(z) = C(1_Y, J_t)(z)$. For any $z = \big[y_i, [x_i, s_i]\big]$ in U, $J_1([x_i, s_i])$ will be $*$ for at least one index i, and so $J_1(z) \in F_{j-1}C(Y, \Sigma X)$. It is clear that J_t preserves $F_{j-1}C(Y, \Sigma X)$ and so part (2a) is complete.

If $(w, s) \in q^{-1}(U)$ and $w = \big[a_i, b_i, y_i, x_i\big]_{1 \leq i \leq j}$, define

$$H_t(w, s) = \Big(\big[(1-t)a_i + ta_i', (1-t)b_i + tb_i', y_i, K_t(x_i)\big]_{1 \leq i \leq j},\ s\Big),$$

where $a_i' = a_i + \frac{1}{4}(b_i - a_i)$ and $b_i' = b_i - \frac{1}{4}(b_i - a_i)$. It is straightforward to verify that $qH_t = h_t q$ and so (2b.) is complete.

Finally, the restriction of H_1 to fibers fits into a homotopy-commutative diagram

$$\begin{tikzcd} q^{-1}(z) \arrow[r, "H_1"] \arrow[d, "\bar{\psi}"'] & q^{-1}(h_1(z)) \arrow[d, "\bar{\psi}"] \\ \bar{C}_1(Y, X) \arrow[r, "\xi \circ C_1(1_Y, K_1)"'] & \bar{C}_1(Y, X) \end{tikzcd}$$

where we have already shown that the maps $\bar{\psi}$ are homotopy equivalences, and where ξ is multiplication by the element

$$\big[a_i', b_i', y_i, K_1(x_i)\big]_{1 \leq i \leq j} \text{ and } {\scriptstyle b_i' \leq s < b_i}$$

in the H-space structure on $\bar{C}_1(Y, X)$. Since $\bar{C}_1(Y, X)$ is connected (because X is) this is a homotopy equivalence. This completes the proof of (2c.), and hence q is a quasifibration.

More can be said. By extending and iterating the definition and theorem, we can prove

Corollary 1 *Let Y be a weak metric space and X a nondegenerately based space. For each $n \geq 1$ there is a space $C_n(Y, X)$ and a pair of maps*

$$C(\mathbb{R}^n \times Y, X) \xleftarrow{\phi_n} C_n(Y, X) \xrightarrow{\alpha_n} \Omega^n C(Y, \Sigma^n X)$$

such that:

1. *$C_n(-,-)$ is functorial with respect to based maps in the second variable and injective maps in the first variable, and ϕ_n and α_n are natural;*
2. *ϕ_n is a homotopy equivalence; and*

3. α_n is a weak homotopy equivalence if X is path-connected.

There is an evident action of the little n-cubes operad $\mathcal{C}_n$ of [1] on all the spaces appearing in the Corollary, and ϕ_n and α_n can be seen to be $\mathcal{C}_n$-maps.

It is also true (and proved in [4]) that when X is not path connected, α_n is a group-completion for $n \geq 2$.

References

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